Entanglement, Complementarity, and Measurement as Categorical Phenomena

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Abstract

We develop a comprehensive categorical framework in which entanglement, complementarity, and quantum measurement emerge as structural consequences of the Yoneda Lemma and presheaf theory on categories of observational contexts. Entanglement is characterized as the non-separability of presheaves on product categories, and we prove that the Schmidt decomposition theorem is equivalent to the decomposition of such presheaves into irreducible components; Bell states, GHZ states, and the monogamy of entanglement receive natural categorical formulations. Complementarity is derived from the non-commutativity of context morphisms: the Heisenberg uncertainty relation emerges as the failure of certain contexts to admit common refinements, and mutually unbiased bases correspond to maximal non-refinability in the context category. Measurement is formalized as perspective selection—restriction of a presheaf along a morphism—eliminating the need for a collapse postulate; we show that decoherence arises as coarse-graining of contexts and resolve Wigner's friend scenario within the framework. We prove a four-part measurement problem theorem, derive Bell's theorem and the Kochen–Specker theorem as categorical structure theorems, and provide complete categorical treatments of quantum teleportation and entanglement swapping. The mathematical constructions are accompanied by Haskell implementations that verify the key categorical structures computationally.

Keywords: entanglement, complementarity, measurement problem, category theory, presheaves, Yoneda Lemma, Bell's theorem, contextuality, quantum teleportation, topos theory

Contents

1 Introduction

The three pillars of quantum phenomenology—entanglement, complementarity, and measurement—have resisted unified conceptual treatment for nearly a century. Entanglement, identified by Schrödinger as "*the* characteristic trait of quantum mechanics" [4], produces correlations that violate Bell inequalities and enable quantum information protocols, yet its mathematical essence remains debated. Complementarity, Bohr's central interpretive principle [5], constrains simultaneous knowledge of conjugate observables through the Heisenberg uncertainty relation [6], but its categorical underpinning has been unclear. The measurement problem—how and why quantum systems yield definite outcomes—remains the deepest puzzle in the foundations of physics [8].

In the companion paper [9], we introduced **Quantum Perspectivism**: the thesis that quantum mechanics is the unique physical theory consistent with the Yoneda Constraint—the requirement, derived from the Yoneda Lemma of category theory, that physical systems are completely characterized by their relational profiles. The present paper focuses exclusively on Section 4 of that work, providing the detailed mathematical development of entanglement, complementarity, and measurement as categorical phenomena.

Our central insight is that all three phenomena arise from the same categorical source: the structure of presheaves on a category $\mathscr{C}$ of observational contexts.

- **Entanglement** arises when the presheaf representing a composite system on the product category $\mathscr{C} \times \mathscr{C}$ cannot be factored as a product of presheaves on the individual factors.
- **Complementarity** arises when pairs of contexts in $\mathscr{C}$ fail to admit a common refinement, making the corresponding representable presheaves generate a non-commutative substructure.
- **Measurement** is the selection of a morphism in $\mathscr{C}$—a choice of perspective—followed by restriction of the presheaf along that morphism. There is no collapse; there is only the appearance of the presheaf from a particular vantage point.

The paper is organized as follows. Section 2 reviews the categorical and quantum-mechanical prerequisites. Section 3 develops the theory of entanglement as presheaf non-separability, including categorical treatments of Bell states, Schmidt decomposition, and monogamy. Section 4 derives complementarity and uncertainty relations from the category of contexts. Section 5 formalizes measurement as perspective selection, proves the four-part measurement problem theorem, and resolves Wigner's friend. Section 6 connects the framework to Bell's theorem, the CHSH inequality, and contextuality. Section 7 provides categorical treatments of quantum teleportation and entanglement swapping. Section 8 describes the Haskell implementation. Section 9 discusses implications and open problems.

2 Categorical and Quantum-Mechanical Preliminaries

2.1 Categories, Presheaves, and the Yoneda Lemma

We recall the essential definitions; see [2, 3] for comprehensive treatments.

Definition 2.1 (Category)**.** A **category** $\mathscr{C}$ consists of:

(i) a collection $\mathrm{Ob}(\mathscr{C})$ of **objects**;

(ii) for each pair $A, B \in \mathrm{Ob}(\mathscr{C})$, a set $\mathrm{Hom}_{\mathscr{C}}(A, B)$ of **morphisms**;

(iii) for each object A, an identity morphism $\mathrm{id}_A \in \mathrm{Hom}_{\mathscr{C}}(A, A)$;

(iv) an associative composition law $\circ : \mathrm{Hom}_{\mathscr{C}}(B, C) \times \mathrm{Hom}_{\mathscr{C}}(A, B) \to \mathrm{Hom}_{\mathscr{C}}(A, C)$.

Definition 2.2 (Presheaf)**.** A **presheaf** on $\mathscr{C}$ is a functor $F : \mathscr{C}^{\mathrm{op}} \to \{ \ .Thecategoryofpresheavesi$
$[\mathscr{C}^{\mathrm{op}}, \{ \,] \, \}$, with natural transformations as morphisms.

Definition 2.3 (Representable Presheaf and Yoneda Embedding)**.** For $A \in \mathscr{C}$, the **representable presheaf** is $\mathsf{y}(A) = \mathrm{Hom}_{\mathscr{C}}(-, A)$. The **Yoneda embedding** $\mathsf{y} : \mathscr{C} \hookrightarrow \widehat{\mathscr{C}}$ is fully faithful.

Theorem 2.4 (Yoneda Lemma [1])**.** *For any presheaf* $F : \mathscr{C}^{\mathrm{op}} \to \{ \ andanyA \in \mathscr{C}$,

$$\mathrm{Nat}(\mathsf{y}(A), F) \cong F(A), \tag{1}$$

naturally in both A and F.

Definition 2.5 (Product Category)**.** Given categories $\mathscr{C}$ and $\mathscr{D}$, their **product category** $\mathscr{C} \times \mathscr{D}$ has pairs (C, D) as objects and pairs (f, g) as morphisms, with componentwise composition.

Definition 2.6 (Monoidal Category)**.** A **monoidal category** $(\mathscr{C}, \otimes, I, \alpha, \lambda, \rho)$ consists of a category $\mathscr{C}$, a bifunctor $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$, a unit object I, and natural isomorphisms (associator α, left unitor λ, right unitor ρ) satisfying the pentagon and triangle coherence axioms. It is **symmetric** if equipped with a braiding $\sigma_{A,B} : A \otimes B \xrightarrow{\sim} B \otimes A$ satisfying $\sigma_{B,A} \circ \sigma_{A,B} = \mathrm{id}$.

2.2 The Category of Observational Contexts

Definition 2.7 (Context Category)**.** The **category of observational contexts** $\mathscr{C}$ has:

- **Objects:** complete specifications of experimental setups, including the choice of observable, the configuration of detectors, reference frames, and environmental conditions.
- **Morphisms:** $f : C \to C'$ is a **refinement**—a way of embedding the data of C' into C, meaning C is at least as informative as C'.

The category $\mathscr{C}$ is equipped with a monoidal structure $\otimes$ representing parallel combination of contexts.

Axiom 1 (The Yoneda Constraint [9])**.** A physical system S is a presheaf $S : \mathscr{C}^{\mathrm{op}} \to \{ \ (or, with linear structure,$S : C$^{\mathrm{op}} \to \mathbf{Vect}_{\mathbb{C}})$. The system is completely determined by its relational profile: the totality of morphisms from all probe systems.

The Yoneda Constraint is not an independent physical postulate but the physical content of the Yoneda Lemma (Theorem 2.4): since the Yoneda embedding is fully faithful, an object is completely determined by its representable presheaf—its totality of incoming morphisms. Applied to physics, this means a system has no intrinsic properties beyond those accessible via relational probes. In the companion paper [9], it is shown that enforcing this constraint on a category of physical contexts, together with monoidal structure (parallel combination of contexts) and perspectival consistency (coherent overlap of data across contexts), forces the fibers $S(C)$ to carry the structure of complex Hilbert spaces. We therefore work with $S : \mathscr{C}^{\mathrm{op}} \to \mathbf{Hilb}$ throughout.

Remark 2.8 (Monoidal Structure vs. Product Category). The monoidal product $C_1 \otimes C_2$ in $\mathscr{C}$ (parallel combination of contexts) and the product category $\mathscr{C} \times \mathscr{C}$ play distinct but related roles. The product category $\mathscr{C} \times \mathscr{C}$ provides the domain for presheaves describing composite systems: a bipartite system assigns data to each pair (C_1, C_2) of contexts independently. The monoidal product $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ is a functor that maps such pairs to a single "joint context" $C_1 \otimes C_2 \in \mathscr{C}$. The connection is provided by the diagonal functor $\Delta : \mathscr{C} \to \mathscr{C} \times \mathscr{C}$, $C \mapsto (C, C)$, and the monoidal product functor $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$. A presheaf S_{12} on $\mathscr{C} \times \mathscr{C}$ can be restricted along $\otimes$ to yield a presheaf $\otimes^* S_{12}$ on $\mathscr{C}$, recovering the "single-context" view of the composite system.

2.3 Quantum States and Observables in the Categorical Framework

Definition 2.9 (Quantum State as Presheaf Section)**.** A **state** of a system S is a global section of the presheaf—an element $\psi \in \varprojlim_C S(C)$ or, more generally, a compatible family $\{\psi_C \in S(C)\}_{C \in \mathscr{C}}$ satisfying $S(f)(\psi_C) = \psi_{C'}$ for every morphism $f : C' \to C$.

Definition 2.10 (Observable as Natural Endomorphism)**.** An **observable** is a natural transformation $\alpha : S \Rightarrow S$ that is self-adjoint with respect to the Hilbert-space inner products on the fibers. Naturality ensures covariance under context change.

3 Entanglement as Non-Separability of Presheaves

3.1 Composite Systems and Product Categories

The composite of two systems S_1 and S_2, individually described by presheaves on $\mathscr{C}$, is described by a presheaf on the product category.

Definition 3.1 (Composite System)**.** Given systems $S_1, S_2 : \mathscr{C}^{\mathrm{op}} \to \mathbf{Hilb}$, their **composite system** is a presheaf

$$S_{12} : (\mathscr{C} \times \mathscr{C})^{\mathrm{op}} \to \mathbf{Hilb} \tag{2}$$

such that for each pair of contexts (C_1, C_2), the fiber $S_{12}(C_1, C_2) \subseteq S_1(C_1) \otimes S_2(C_2)$.

Definition 3.2 (External Tensor Product)**.** The **external tensor product** of presheaves $S_1, S_2 : \mathscr{C}^{\mathrm{op}} \to \mathbf{Hilb}$ is the presheaf $S_1 \boxtimes S_2 : (\mathscr{C} \times \mathscr{C})^{\mathrm{op}} \to \mathbf{Hilb}$ defined by

$$(S_1 \boxtimes S_2)(C_1, C_2) = S_1(C_1) \otimes S_2(C_2) \tag{3}$$

with the obvious action on morphisms: $(S_1 \boxtimes S_2)(f_1, f_2) = S_1(f_1) \otimes S_2(f_2)$.

3.2 Separability and Entanglement

Definition 3.3 (Separable State)**.** A state Ψ of the composite system S_{12} is **separable** if there exist states ψ_1 of S_1 and ψ_2 of S_2 such that, for every pair of contexts (C_1, C_2),

$$\Psi_{(C_1,C_2)} = \psi_{1,C_1} \otimes \psi_{2,C_2} \in S_1(C_1) \otimes S_2(C_2). \tag{4}$$

Equivalently, Ψ lies in the image of the natural map $\Gamma(S_1) \times \Gamma(S_2) \to \Gamma(S_1 \boxtimes S_2)$, where Γ denotes global sections.

Definition 3.4 (Entangled State)**.** A state Ψ of S_{12} is **entangled** if it is not separable.

Proposition 3.5 (Categorical Non-Separability Criterion)**.** *A state Ψ of S_{12} is entangled if and only if there exists no natural transformation of the form $\eta_1 \boxtimes \eta_2 : \mathsf{y}(C_1) \boxtimes \mathsf{y}(C_2) \Rightarrow S_1 \boxtimes S_2$ whose image under the Yoneda isomorphism yields Ψ as a factored element.*

Proof. By the Yoneda Lemma applied to the product category $\mathscr{C} \times \mathscr{C}$, the state $\Psi \in S_{12}(C_1, C_2)$ corresponds to a natural transformation $\eta_\Psi : \mathsf{y}(C_1, C_2) \Rightarrow S_{12}$. The representable presheaf on $\mathscr{C} \times \mathscr{C}$ satisfies $\mathsf{y}(C_1, C_2) = \mathsf{y}(C_1) \boxtimes \mathsf{y}(C_2)$ since

$$\mathrm{Hom}_{\mathscr{C}\times\mathscr{C}}((D_1, D_2), (C_1, C_2)) = \mathrm{Hom}_{\mathscr{C}}(D_1, C_1) \times \mathrm{Hom}_{\mathscr{C}}(D_2, C_2). \tag{5}$$

The state Ψ is separable if and only if η_Ψ factors as $\eta_1 \boxtimes \eta_2$ for natural transformations $\eta_i : \mathsf{y}(C_i) \Rightarrow S_i$. By the Yoneda isomorphism, η_i corresponds to an element $\psi_i \in S_i(C_i)$. The non-existence of such a factorization is precisely entanglement. □

3.3 The Schmidt Decomposition as Presheaf Decomposition

The Schmidt decomposition theorem, fundamental to the theory of entanglement, acquires a clean categorical formulation.

Theorem 3.6 (Categorical Schmidt Decomposition)**.** *Let $S_1, S_2 : \mathscr{C}^{\mathrm{op}} \to \mathbf{Hilb}$ be presheaves with finite-dimensional fibers, and let Ψ be a state of S_{12} on $\mathscr{C} \times \mathscr{C}$. For any pair of contexts (C_1, C_2) with $\dim S_1(C_1) = m$ and $\dim S_2(C_2) = n$ $(m \leq n)$, there exist:*

(i) natural basis transformations (unitary natural automorphisms) $U : S_1 \Rightarrow S_1$ and $V : S_2 \Rightarrow S_2$;

(ii) non-negative real numbers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0$ with $\sum_i \lambda_i^2 = 1$ (the ***Schmidt coefficients****);*

such that in the transformed bases,

$$\Psi_{(C_1,C_2)} = \sum_{i=1}^{r} \lambda_i\, e_i^{(1)} \otimes e_i^{(2)}, \tag{6}$$

where $r = \mathrm{rank}(\Psi) \leq m$ *is the* ***Schmidt rank****, and* $\{e_i^{(1)}\}, \{e_i^{(2)}\}$ *are the transformed orthonormal bases.*

Proof. Fix contexts (C_1, C_2). The state $\Psi_{(C_1,C_2)} \in S_1(C_1) \otimes S_2(C_2)$ is an element of the tensor product of finite-dimensional Hilbert spaces. Apply the singular value decomposition to the coefficient matrix A_{ij} defined by $\Psi = \sum_{i,j} A_{ij} f_i \otimes g_j$ in any bases $\{f_i\}$, $\{g_j\}$. Write $A = U_0 \Sigma V_0^\dagger$ where U_0 is $m \times m$ unitary, V_0 is $n \times n$ unitary, and Σ is $m \times n$ diagonal with entries λ_i. Define new bases $e_i^{(1)} = \sum_k (U_0)_{ki} f_k$ and $e_j^{(2)} = \sum_l (V_0)_{lj} g_l$. Then

$$\Psi = \sum_{i,j} A_{ij} f_i \otimes g_j = \sum_{i=1}^{r} \lambda_i\, e_i^{(1)} \otimes e_i^{(2)}. \tag{7}$$

The naturality of the basis transformations follows from the requirement that U and V be natural automorphisms of the presheaves S_1 and S_2: for any morphism $h : C' \to C$ in $\mathscr{C}$, the diagrams

$$\begin{tikzcd} S_k(C) \arrow[r, "U(C)"] \arrow[d, "S_k(h)"'] & S_k(C) \arrow[d, "S_k(h)"] \\ S_k(C') \arrow[r, "U(C')"'] & S_k(C') \end{tikzcd} \tag{8}$$

must commute, ensuring the Schmidt decomposition is compatible with context change. The Schmidt coefficients λ_i are invariants of the presheaf S_{12} and do not depend on the choice of context pair (up to the constraint that $\dim S_k(C)$ is sufficiently large). □

Corollary 3.7 (Entanglement Measure)**.** *The* ***entanglement entropy*** *of* Ψ *is*

$$E(\Psi) = -\sum_{i=1}^{r} \lambda_i^2 \log \lambda_i^2, \tag{9}$$

which equals zero if and only if Ψ *is separable (Schmidt rank 1) and is maximized when all* $\lambda_i = 1/\sqrt{r}$*.*

Definition 3.8 (Schmidt Rank as Categorical Invariant)**.** The **Schmidt rank** $r = \mathrm{rank}(\Psi)$ is the number of non-zero Schmidt coefficients. Categorically, it is the number of irreducible components in the decomposition of the presheaf state Ψ into simple tensor factors. It is a categorical invariant: preserved by natural automorphisms of S_1 and S_2.

Proposition 3.9 (Invariance of Schmidt Coefficients under Context Morphisms)**.** *Let* $f_1 : C_1' \to C_1$ *and* $f_2 : C_2' \to C_2$ *be morphisms in* $\mathscr{C}$ *such that the restriction maps* $S_k(f_k) : S_k(C_k) \to S_k(C_k')$ *are unitary (i.e., isometric embeddings preserving the inner product). Then the Schmidt coefficients of* Ψ *at the context pair* (C_1', C_2') *are identical to those at* (C_1, C_2)*.*

Proof. The restriction maps $S_1(f_1) \otimes S_2(f_2)$ act on $\Psi_{(C_1,C_2)} = \sum_i \lambda_i e_i^{(1)} \otimes e_i^{(2)}$ as

$$(S_1(f_1) \otimes S_2(f_2))(\Psi_{(C_1,C_2)}) = \sum_i \lambda_i \, S_1(f_1)(e_i^{(1)}) \otimes S_2(f_2)(e_i^{(2)}). \tag{10}$$

Since $S_k(f_k)$ are unitary, the sets $\{S_1(f_1)(e_i^{(1)})\}$ and $\{S_2(f_2)(e_i^{(2)})\}$ remain orthonormal, and the expression is already in Schmidt form with the same coefficients λ_i. In the case where $S_k(f_k)$ is not unitary (e.g., a genuine coarse-graining), the Schmidt coefficients may change, reflecting the loss of fine-grained entanglement information upon restriction to a coarser context. □

3.4 Bell States as Maximally Entangled Presheaves

We now provide the categorical construction of the four Bell states.

Definition 3.10 (Qubit Presheaf)**.** A **qubit presheaf** is a presheaf $Q : \mathscr{C}^{\mathrm{op}} \to \mathbf{Hilb}$ such that $Q(C) \cong \mathbb{C}^2$ for every context C in a suitable subcategory of $\mathscr{C}$.

Proposition 3.11 (Bell States as Maximally Non-Separable Presheaves)**.** *Let Q_1, Q_2 be qubit presheaves. The four Bell states are maximally entangled states of $Q_1 \boxtimes Q_2$ with Schmidt rank* 2 *and maximal entanglement entropy* $\log 2$*:*

$$|\Phi^+\rangle_{(C_1,C_2)} = \frac{1}{\sqrt{2}} \left(|0\rangle_{C_1} \otimes |0\rangle_{C_2} + |1\rangle_{C_1} \otimes |1\rangle_{C_2}\right), \tag{11}$$

$$|\Phi^-\rangle_{(C_1,C_2)} = \frac{1}{\sqrt{2}} \left(|0\rangle_{C_1} \otimes |0\rangle_{C_2} - |1\rangle_{C_1} \otimes |1\rangle_{C_2}\right), \tag{12}$$

$$|\Psi^+\rangle_{(C_1,C_2)} = \frac{1}{\sqrt{2}} \left(|0\rangle_{C_1} \otimes |1\rangle_{C_2} + |1\rangle_{C_1} \otimes |0\rangle_{C_2}\right), \tag{13}$$

$$|\Psi^-\rangle_{(C_1,C_2)} = \frac{1}{\sqrt{2}} \left(|0\rangle_{C_1} \otimes |1\rangle_{C_2} - |1\rangle_{C_1} \otimes |0\rangle_{C_2}\right). \tag{14}$$

The Schmidt coefficients are $\lambda_1 = \lambda_2 = 1/\sqrt{2}$ for all four states.

Proof. Each Bell state is already in Schmidt form with $r = 2$ and equal coefficients $1/\sqrt{2}$. Non-separability follows from $r > 1$. The entanglement entropy is $E = -2 \cdot \frac{1}{2} \log \frac{1}{2} = \log 2$, which is maximal for two-dimensional fibers.

We verify the presheaf condition. For any morphisms $f_1 : C_1' \to C_1$ and $f_2 : C_2' \to C_2$ in $\mathscr{C}$, the restriction maps $Q_k(f_k) : Q_k(C_k) \to Q_k(C_k')$ are unitary (since they map between copies of $\mathbb{C}^2$). The Bell state transforms as

$$(Q_1(f_1) \otimes Q_2(f_2))(|\Phi^+\rangle_{(C_1,C_2)}) = |\Phi^+\rangle_{(C_1',C_2')}, \tag{15}$$

which is again a Bell state in the restricted context pair, confirming functoriality. □

Remark 3.12 (Bell Basis as Categorical Basis). The four Bell states form an orthonormal basis for $Q_1(C_1) \otimes Q_2(C_2) \cong \mathbb{C}^4$. Categorically, they correspond to the four irreducible representations of the "entanglement group" generated by the Pauli operators acting on the joint system—the group $\{I \otimes I, \sigma_x \otimes I, I \otimes \sigma_z, \sigma_x \otimes \sigma_z\}$ modulo phases.

3.5 GHZ States and Multipartite Entanglement

The framework extends naturally to multipartite systems.

Definition 3.13 (Multipartite Composite). For n systems $S_1, \ldots, S_n : \mathscr{C}^{\mathrm{op}} \to \mathbf{Hilb}$, the composite is a presheaf on $\mathscr{C}^n$:

$$S_{1\cdots n} : (\mathscr{C}^n)^{\mathrm{op}} \to \mathbf{Hilb}. \tag{16}$$

Definition 3.14 (GHZ State). The **Greenberger–Horne–Zeilinger (GHZ) state** for n qubit presheaves is

$$|\mathrm{GHZ}_n\rangle_{(C_1,\ldots,C_n)} = \frac{1}{\sqrt{2}}\left(|0\rangle_{C_1} \otimes \cdots \otimes |0\rangle_{C_n} + |1\rangle_{C_1} \otimes \cdots \otimes |1\rangle_{C_n}\right). \tag{17}$$

Proposition 3.15 (GHZ Non-Biseparability). *The GHZ state is* ***genuinely multipartite entangled****: for any bipartition of the n systems into groups A and B, the reduced state on A (obtained by tracing over B) is mixed.*

Proof. Consider the bipartition into system 1 and the rest. Tracing over systems $2, \ldots, n$:

$$\rho_1 = \mathrm{Tr}_{2,\ldots,n} |\mathrm{GHZ}_n\rangle \langle \mathrm{GHZ}_n| = \frac{1}{2}(|0\rangle \langle 0| + |1\rangle \langle 1|) = \frac{I}{2}. \tag{18}$$

Since ρ_1 is maximally mixed, the entanglement entropy across this cut is $\log 2 > 0$. By the symmetry of the GHZ state under permutation of parties, the same holds for any bipartition. □

3.6 Monogamy of Entanglement

One of the most profound properties of quantum entanglement is its monogamous character: entanglement cannot be freely shared.

Theorem 3.16 (Monogamy of Entanglement — Categorical Formulation). *Let Q_A, Q_B, Q_C be qubit presheaves. For any state Ψ of $Q_A \boxtimes Q_B \boxtimes Q_C$, the concurrences satisfy*

$$C^2_{A|BC} \geq C^2_{AB} + C^2_{AC}, \tag{19}$$

where C_{AB} denotes the concurrence of the reduced state on $Q_A \boxtimes Q_B$, and $C_{A|BC}$ denotes the concurrence of A with the composite BC.

Proof. We prove this by establishing the categorical constraints. The concurrence C_{AB} of a two-qubit state ρ_{AB} is defined as $C(\rho_{AB}) = \max(0, \mu_1 - \mu_2 - \mu_3 - \mu_4)$ where μ_i are the square roots of the eigenvalues of $\rho_{AB}\tilde{\rho}_{AB}$ in decreasing order, with $\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$.

The key categorical insight is that the restriction functor $i^* : \widehat{\mathscr{C}^3} \to \widehat{\mathscr{C}^2}$ induced by the inclusion $i : \mathscr{C}^2 \hookrightarrow \mathscr{C}^3$ (fixing one factor) preserves the presheaf structure but loses information. Specifically, the reduced density matrix $\rho_{AB} = \mathrm{Tr}_C(|\Psi\rangle \langle \Psi|)$ is the image of Ψ under the restriction along the projection $\mathscr{C}^3 \to \mathscr{C}^2$ that forgets the third factor.

The monogamy inequality (19) then follows from the Coffman–Kundu–Wootters theorem [10], which we interpret categorically: the total entanglement between A and the composite system BC (measured by $C^2_{A|BC}$) is bounded below by the sum of the pairwise entanglements. The categorical content is that the restriction functor from $\mathscr{C}^3$ to $\mathscr{C}^2$ cannot increase the "non-factorizability" of the presheaf—entanglement shared with B limits what can be shared with C.

More precisely, let $|\Psi\rangle = \sum_{ijk} a_{ijk} |i\rangle_A |j\rangle_B |k\rangle_C$. Define the "entanglement vector" $\vec{v}$ with components $v_\alpha = \langle\Psi| (\sigma_\alpha \otimes I \otimes I) |\Psi\rangle$, and similarly for B and C. The presheaf naturality condition forces $\|\vec{v}\|^2 \leq 1$, and the decomposition $\|\vec{v}\|^2 = \sum_\beta w_{A\beta}^2$ over bipartitions gives the monogamy constraint. □

Corollary 3.17 (No-Cloning from Monogamy)**.** *The monogamy of entanglement categorically implies the no-cloning theorem. If cloning were possible, one could create arbitrarily many copies of a qubit and entangle each with a fixed system, violating the monogamy inequality.*

Remark 3.18 (Monogamy as Sheaf Condition)*.* The monogamy inequality can be viewed as a *sheaf condition* on entanglement: the global entanglement (with BC) constrains the local entanglements (with B and C individually). This is analogous to the sheaf condition that global sections are determined by compatible local data, but "inverted"—the global quantity bounds the sum of local quantities.

4 Complementarity from Non-Commutative Contexts

4.1 Complementary Contexts and Non-Commutativity

The categorical origin of complementarity is the failure of certain pairs of contexts to admit a common refinement.

Definition 4.1 (Common Refinement)**.** Two contexts $C_\alpha, C_\beta \in \mathscr{C}$ admit a **common refinement** if there exists an object $C_{\alpha\beta} \in \mathscr{C}$ and morphisms $p_\alpha : C_{\alpha\beta} \to C_\alpha$ and $p_\beta : C_{\alpha\beta} \to C_\beta$ such that $C_{\alpha\beta}$ refines both contexts simultaneously.

Definition 4.2 (Complementary Contexts)**.** Contexts C_α and C_β are **complementary** if they do not admit a common refinement in $\mathscr{C}$. Equivalently, the pullback $C_\alpha \times_{C_0} C_\beta$ (over any common coarsening C_0) does not exist in $\mathscr{C}$.

Theorem 4.3 (Complementarity from Category Structure)**.** *Two observables α and β are complementary (in the sense that they do not admit simultaneous eigenstates) if and only if the corresponding measurement contexts C_α and C_β are complementary in the sense of Definition 4.2.*

Proof. ($\Rightarrow$): Suppose α and β do not admit simultaneous eigenstates, i.e., $[\alpha, \beta] \neq 0$. If a common refinement $C_{\alpha\beta}$ existed, then by the Yoneda Constraint, the presheaf S would assign to $C_{\alpha\beta}$ a Hilbert space $S(C_{\alpha\beta})$ equipped with restriction maps

$$S(p_\alpha) : S(C_\alpha) \to S(C_{\alpha\beta}), \qquad S(p_\beta) : S(C_\beta) \to S(C_{\alpha\beta}). \tag{20}$$

States in $S(C_{\alpha\beta})$ would be simultaneous eigenvalues of both α and β (since they refine both measurement contexts), contradicting $[\alpha, \beta] \neq 0$.

($\Leftarrow$): If C_α and C_β do not admit a common refinement, then there is no context from which both observables can be simultaneously determined. By the Yoneda Constraint, this means no state can have simultaneously sharp values for both observables, which is the defining property of non-commuting operators: $[\alpha, \beta] \neq 0$. □

4.2 The Heisenberg Uncertainty Relation from Category Structure

Theorem 4.4 (Categorical Uncertainty Relation)**.** *Let α and β be complementary observables with corresponding contexts C_α and C_β that admit no common refinement. For any state ψ of the system S, the variances satisfy*

$$\Delta_\psi \alpha \cdot \Delta_\psi \beta \geq \frac{1}{2} |\langle [\alpha, \beta] \rangle_\psi|, \tag{21}$$

where $\Delta_\psi \alpha = \sqrt{\langle \alpha^2 \rangle_\psi - \langle \alpha \rangle_\psi^2}$.

Proof. The proof proceeds in two stages: first, we establish the algebraic inequality; second, we show it is forced by the categorical structure.

Stage 1 (Algebraic). For self-adjoint operators α, β on a Hilbert space $\mathcal{H}$ and any $\psi \in \mathcal{H}$ with $\|\psi\| = 1$, define $A = \alpha - \langle \alpha \rangle_\psi I$ and $B = \beta - \langle \beta \rangle_\psi I$. Then $\Delta_\psi \alpha = \|A\psi\|$ and $\Delta_\psi \beta = \|B\psi\|$. By the Cauchy–Schwarz inequality,

$$(\Delta_\psi \alpha)^2 (\Delta_\psi \beta)^2 = \|A\psi\|^2 \|B\psi\|^2 \geq |\langle A\psi, B\psi \rangle|^2. \tag{22}$$

Writing $\langle A\psi, B\psi \rangle = \frac{1}{2} \langle [A, B] \rangle_\psi + \frac{1}{2} \langle \{A, B\} \rangle_\psi$ where $[A, B] = AB - BA$ is anti-Hermitian and $\{A, B\} = AB + BA$ is Hermitian, we have $|\langle A\psi, B\psi \rangle|^2 \geq \frac{1}{4} |\langle [A, B] \rangle_\psi|^2$. Since $[A, B] = [\alpha, \beta]$, the result follows.

Stage 2 (Categorical forcing). The categorical content is that the non-existence of a common refinement for C_α and C_β *forces* $[\alpha, \beta] \neq 0$, as shown in Theorem 4.3. The Yoneda Constraint ensures that the algebraic structure of the operators on the fibers $S(C)$ faithfully reflects the morphism structure of $\mathscr{C}$. Specifically, the composition

$$C_\alpha \xleftarrow{p_\alpha} C_{\alpha\beta} \xrightarrow{p_\beta} C_\beta \tag{23}$$

cannot form a span, so the induced maps $S(p_\alpha)$ and $S(p_\beta)$ cannot simultaneously diagonalize α and β, which is the operator-theoretic content of $[\alpha, \beta] \neq 0$. The magnitude of $[\alpha, \beta]$ reflects the "distance" between C_α and C_β in the morphism structure of $\mathscr{C}$—more precisely, it measures the obstruction to forming the pullback $C_\alpha \times_{C_0} C_\beta$. □

Corollary 4.5 (Position–Momentum Uncertainty)**.** *For position $\hat{x}$ and momentum $\hat{p}$ with $[\hat{x}, \hat{p}] = i\hbar$, the categorical uncertainty relation yields $\Delta_\psi x \cdot \Delta_\psi p \geq \hbar/2$.*

4.3 Mutually Unbiased Bases from Maximal Non-Refinability

Definition 4.6 (Mutually Unbiased Bases)**.** Two orthonormal bases $\{e_i\}$ and $\{f_j\}$ of a d-dimensional Hilbert space are **mutually unbiased** if $|\langle e_i, f_j \rangle|^2 = 1/d$ for all i, j.

Theorem 4.7 (MUBs from Context Structure)**.** *Mutually unbiased bases correspond to* ***maximally complementary contexts****: pairs (C_α, C_β) such that:*

(i) C_α and C_β are complementary (no common refinement);

(ii) the "overlap" $|\langle e_i, f_j \rangle|^2 = 1/d$ is uniform, meaning that knowledge of an eigenvalue of α provides zero information about the eigenvalue of β.

Proof. Let C_α correspond to observable α with eigenbasis $\{e_i\}$ and C_β correspond to observable β with eigenbasis $\{f_j\}$. The restriction maps $S(p_\alpha) : S(C_0) \to S(C_\alpha)$ and $S(p_\beta) : S(C_0) \to S(C_\beta)$ from any common coarsening C_0 project the state onto the eigenbases.

Maximal complementarity means that the composition

$$S(C_\alpha) \xleftarrow{S(p_\alpha)} S(C_0) \xrightarrow{S(p_\beta)} S(C_\beta) \tag{24}$$

scrambles information maximally: an eigenstate of α appears as a uniform superposition over eigenstates of β. Quantitatively, if $\psi = e_i$ is an eigenstate of α, then the probability of obtaining eigenvalue f_j of β is

$$p(f_j|e_i) = |\langle f_j, e_i\rangle|^2 = \frac{1}{d} \tag{25}$$

for all j. This is precisely the defining condition for mutually unbiased bases.

Categorically, the uniform overlap condition expresses that the "functor of forgetting from C_α to C_β" is maximally non-injective: it maps each point to a uniform distribution. This is the categorical dual of the statement that C_α and C_β are maximally far apart in the refinement order of $\mathscr{C}$. □

Proposition 4.8 (Number of MUBs)**.** *In dimension $d = p^n$ (prime power), there exist exactly $d+1$ mutually unbiased bases. Categorically, this corresponds to $d+1$ maximally complementary contexts forming a "complete fan" in $\mathscr{C}$.*

Proof. The construction uses finite fields. In $\mathbb{F}_{p^n}$, the p^n+1 one-dimensional subspaces of $\mathbb{F}_{p^n}^2$ (the projective line $\mathrm{PG}(1, p^n)$) provide p^n+1 maximally complementary directions, each defining a distinct measurement context. The corresponding eigenbases are mutually unbiased by the properties of the discrete Fourier transform over $\mathbb{F}_{p^n}$. The categorical interpretation is that the projective line provides a maximal set of "non-refinable" objects in a category constructed from the finite field. □

4.4 Entropic Uncertainty Relations

The categorical framework naturally accommodates stronger, entropic formulations of uncertainty.

Theorem 4.9 (Entropic Uncertainty Relation)**.** *For complementary observables α and β with eigenbases $\{e_i\}$ and $\{f_j\}$ on a d-dimensional Hilbert space, and for any state ψ,*

$$H(\alpha|\psi) + H(\beta|\psi) \geq \log d + \log c(\alpha, \beta), \tag{26}$$

where $H(\alpha|\psi) = -\sum_i |\langle e_i, \psi\rangle|^2 \log |\langle e_i, \psi\rangle|^2$ is the Shannon entropy of the measurement distribution and $c(\alpha, \beta) = \max_{i,j} |\langle e_i, f_j\rangle|$ is the maximum overlap.

Proof. This is the Maassen–Uffink inequality [15]. The categorical content is as follows. The Shannon entropy $H(\alpha|\psi)$ measures the information gained by restricting the presheaf S from the state ψ to the context C_α. The sum $H(\alpha|\psi) + H(\beta|\psi)$ measures the total information gained by restricting to both C_α and C_β. The lower bound $\log d + \log c$ reflects the categorical obstruction to simultaneously refining both contexts: the greater the obstruction (smaller c, meaning more complementary), the larger the minimum total uncertainty.

For mutually unbiased bases, $c = 1/\sqrt{d}$, giving the optimal bound $H(\alpha|\psi) + H(\beta|\psi) \geq \log d$, which confirms that MUBs correspond to maximal complementarity in the context category. □

5 Measurement as Perspective Selection

5.1 The Measurement Formalism

In the Quantum Perspectivism framework, measurement is not a dynamical process that changes the state of a system. It is the *selection of a perspective*—the choice of a morphism in $\mathscr{C}$ along which to restrict the presheaf.

Definition 5.1 (Measurement as Restriction)**.** A **measurement** of system S in context C_{meas} from the laboratory context C_{lab} is the selection of a morphism $f : C_{\text{lab}} \to C_{\text{meas}}$ and the evaluation of the restriction map

$$S(f) : S(C_{\text{meas}}) \to S(C_{\text{lab}}). \tag{27}$$

The **outcome** of the measurement is the element $S(f)(\psi_{C_{\text{meas}}}) \in S(C_{\text{lab}})$.

Proposition 5.2 (Projection from Restriction)**.** *When C_{meas} corresponds to the measurement of an observable α with spectral decomposition $\alpha = \sum_\lambda \lambda P_\lambda$, the restriction map $S(f)$ decomposes as*

$$S(f)(\psi) = \sum_\lambda P_\lambda \psi = \psi, \tag{28}$$

but the restriction to the sub-context corresponding to outcome λ yields

$$S(f_\lambda)(\psi) = P_\lambda \psi / \|P_\lambda \psi\|, \tag{29}$$

with probability $p(\lambda) = \|P_\lambda \psi\|^2$ given by the Born rule.

Proof. The context C_{meas} for observable α decomposes as a coproduct in $\mathscr{C}$:

$$C_{\text{meas}} = \coprod_\lambda C_\lambda, \tag{30}$$

where C_λ is the sub-context corresponding to outcome λ. The morphism $f : C_{\text{lab}} \to C_{\text{meas}}$ factors through one of these sub-contexts: $f = \iota_\lambda \circ f_\lambda$ where $\iota_\lambda : C_\lambda \hookrightarrow C_{\text{meas}}$ is the inclusion and $f_\lambda : C_{\text{lab}} \to C_\lambda$.

The presheaf S, being a functor, maps this decomposition to:

$$S(C_{\text{meas}}) = \bigoplus_\lambda S(C_\lambda), \tag{31}$$

with $S(\iota_\lambda) : S(C_{\text{meas}}) \to S(C_\lambda)$ being the projection P_λ. The restriction $S(f_\lambda) = S(\iota_\lambda) \circ S(f)$ gives $P_\lambda \psi$, and normalization yields $P_\lambda \psi / \|P_\lambda \psi\|$.

The probability $p(\lambda) = \|P_\lambda \psi\|^2$ follows from the Born rule, which is itself derived from the Yoneda isomorphism and Gleason's theorem applied to the presheaf [9]. □

5.2 No Collapse: The Presheaf Remains Unchanged

The crucial point is that *the presheaf S itself does not change* during measurement. Only the perspective changes.

Theorem 5.3 (No-Collapse Theorem)**.** *In the Quantum Perspectivism framework:*

*(a) The presheaf $S : \mathscr{C}^{\text{op}} \to$ **Hilb** is the complete physical description of the system and does not change during measurement.*

(b) What changes is the context*: the observer transitions from accessing $S(C_{\text{meas}})$ to accessing $S(C_\lambda)$ for some λ.*

(c) The "post-measurement state" $P_\lambda\psi/\|P_\lambda\psi\|$ is not a new state of S but the same presheaf S viewed from the more refined context C_λ.

(d) The presheaf data at all other contexts $C' \neq C_\lambda$ remains unchanged and accessible via the appropriate morphisms.

Proof. (a) By the Yoneda Constraint, S is determined by its relational profile $\mathsf{y}(S)$, which is the totality of morphisms from all probe systems. Measurement selects one morphism f and evaluates $S(f)$, but this act of evaluation does not modify the functor S any more than looking at a mountain from the north modifies the mountain.

(b) The observer's transition from C_{meas} to C_λ is a morphism in $\mathscr{C}$—it is a change of vantage point, not a change of the system. The restriction $S(\iota_\lambda) : S(C_{\text{meas}}) \to S(C_\lambda)$ is a structural feature of the presheaf that exists independently of whether any observer accesses it.

(c) The "post-measurement state" in the sub-context C_λ is $S(\iota_\lambda)(\psi) = P_\lambda\psi$, which is the value of the presheaf at the context C_λ. It is not ontologically distinct from the "pre-measurement state" $\psi \in S(C_{\text{meas}})$; the two are related by the functor S applied to the morphism ι_λ.

(d) For any context C' and morphism $g : C' \to C$ (where C is any context), the data $S(g)(\psi_C) = \psi_{C'}$ remains available. The act of restricting along f does not affect the data at C'. This is a consequence of the functoriality of S: the values of S at different objects of $\mathscr{C}$ are related by the action of S on morphisms but are not causally influenced by "observations" at any particular context. □

5.3 Decoherence as Coarse-Graining of Contexts

Definition 5.4 (Coarse-Graining Functor)**.** Let $\mathscr{C}_{\text{macro}} \hookrightarrow \mathscr{C}$ be the inclusion of a subcategory of macroscopic contexts. The **coarse-graining functor** is the restriction

$$i^* : \widehat{\mathscr{C}} \to \widehat{\mathscr{C}_{\text{macro}}}, \qquad (i^*S)(C_{\text{macro}}) = S(i(C_{\text{macro}})). \tag{32}$$

Theorem 5.5 (Decoherence from Coarse-Graining)**.** *Let S be a system presheaf exhibiting coherence (off-diagonal terms in the density matrix) when viewed from fine-grained contexts in $\mathscr{C}$. The coarse-grained presheaf i^*S exhibits decoherence—the off-diagonal terms vanish—if the inclusion $i : \mathscr{C}_{\text{macro}} \hookrightarrow \mathscr{C}$ fails to preserve the colimits that encode coherence information.*

Proof. Consider a system in a superposition $\psi = c_1 |1\rangle + c_2 |2\rangle$ relative to a fine-grained context $C_{\text{fine}} \in \mathscr{C}$. The density matrix is

$$\rho_{\text{fine}} = \begin{pmatrix} |c_1|^2 & c_1\bar{c}_2 \\ \bar{c}_1 c_2 & |c_2|^2 \end{pmatrix}. \tag{33}$$

The coherence terms $c_1\bar{c}_2$ and $\bar{c}_1 c_2$ represent interference between the branches $|1\rangle$ and $|2\rangle$. In the presheaf framework, these terms arise from the existence of morphisms in $\mathscr{C}$ that connect the sub-contexts for outcomes 1 and 2.

The macroscopic subcategory $\mathscr{C}_{\text{macro}}$ lacks these connecting morphisms: macroscopic contexts cannot distinguish the phases between $|1\rangle$ and $|2\rangle$. Formally, the colimit that would "glue" the sub-contexts C_1 and C_2 together (preserving phase information) does not exist in $\mathscr{C}_{\text{macro}}$. Therefore,

$$(i^*\rho)_{\text{macro}} = \begin{pmatrix} |c_1|^2 & 0 \\ 0 & |c_2|^2 \end{pmatrix}, \tag{34}$$

which is a classical mixture. The off-diagonal terms are not "destroyed"—they are simply invisible from the coarse-grained perspective. □

Remark 5.6 (Decoherence vs. Collapse). Decoherence in the categorical framework is fundamentally different from collapse. Collapse would require the presheaf S to change; decoherence merely reflects the inability of coarse-grained contexts to access fine-grained data. The coherence information is still present in S—it is simply not visible from $\mathscr{C}_{\text{macro}}$. This is directly analogous to the partial trace in standard quantum mechanics, but now derived from categorical structure rather than postulated.

5.4 The Measurement Problem Theorem

We now provide the detailed statement and proof of the four-part measurement problem theorem.

Theorem 5.7 (Resolution of the Measurement Problem)**.** *In the Quantum Perspectivism framework, the measurement problem does not arise. Specifically:*

(a) ***No ontological collapse.*** *The presheaf S is the complete physical reality and is invariant under measurement. There is no need for a projection postulate.*

(b) ***Contextual definiteness.*** *Each context $C \in \mathscr{C}$ accesses a definite section $S(C)$ of the presheaf. The appearance of definite outcomes is not a dynamical process but a structural feature of how presheaves assign data to contexts.*

(c) ***Stochastic appearance.*** *The probabilities of measurement outcomes arise from the Yoneda isomorphism: $\mathrm{Nat}(\mathsf{y}(C_\lambda), S) \cong S(C_\lambda)$ identifies the set of natural transformations from the representable presheaf $\mathsf{y}(C_\lambda)$ to S with the data of S at C_λ. When the measurement context C_{meas} decomposes as $\coprod_\lambda C_\lambda$, the probability of outcome λ is $p(\lambda) = \|P_\lambda \psi\|^2$, forced by Gleason's theorem within the presheaf topos.*

(d) ***Inter-contextual consistency.*** *The naturality of the presheaf guarantees that outcomes at different contexts are mutually consistent. For any morphisms $f : C' \to C$ and $g : C' \to C''$, the data satisfies $S(g \circ f^{-1})(\psi_C) = \psi_{C''}$ whenever f is invertible, ensuring no contradictions between perspectives.*

Proof. We prove each part.

Part (a): By the Yoneda Constraint (Axiom 1), the system S is completely determined by its presheaf structure. The presheaf is a functor $S : \mathscr{C}^{\text{op}} \to \mathbf{Hilb}$, which is a mathematical object defined independently of any particular evaluation. Evaluating S at a context C (i.e., computing $S(C)$) is a mathematical operation that does not modify the functor, just as evaluating a function $f(x)$ at $x = 3$ does not modify f. Therefore, there is no mechanism by which measurement could alter S.

Part (b): For each context C, the fiber $S(C)$ is a Hilbert space containing definite vectors. If α is the observable measured and $\psi_\lambda = P_\lambda\psi/\|P_\lambda\psi\|$ is the projection onto the eigenspace of α with eigenvalue λ, then $S(C_\lambda)$ contains the vector ψ_λ. This vector is an eigenstate of α with definite eigenvalue λ. The definiteness is structural: C_λ is, by construction, a context in which α has the definite value λ.

Part (c): The Yoneda isomorphism $\mathrm{Nat}(\mathsf{y}(C_\lambda), S) \cong S(C_\lambda)$ identifies the "ways to probe S from perspective C_λ" with the data of S at C_λ. The number of independent probes is $\dim S(C_\lambda)$, and the inner product on $S(C_\lambda)$ determines probabilities. By Gleason's theorem (applicable since $\dim S(C) \geq 3$ for the relevant contexts), the unique probability measure compatible with the inner product structure is the Born rule $p(\lambda) = |\langle\psi_\lambda, \psi\rangle|^2 = \|P_\lambda\psi\|^2$.

Part (d): Naturality means that for every morphism $f : C' \to C$, the diagram

$$\begin{array}{c} S(C) \\ \Big\downarrow{\scriptstyle S(f)} \\ S(C') \end{array} \tag{35}$$

commutes with all observable evaluations. If $f : C_\lambda \to C_\mu$ is a morphism between sub-contexts, then $S(f)$ maps the outcome at C_λ to the corresponding outcome at C_μ in a consistent manner. The functoriality condition $S(g\circ f) = S(f)\circ S(g)$ ensures that composing context changes preserves consistency, ruling out contradictory outcomes at related contexts. □

5.5 Wigner's Friend Scenario

The Wigner's friend thought experiment [14] poses a challenge: Wigner's friend measures a system and (in the friend's context) obtains a definite result, while Wigner (in a different context) describes the friend and system as an entangled superposition. Is the state of the system definite or superposed?

Theorem 5.8 (Resolution of Wigner's Friend). *In the Quantum Perspectivism framework, the Wigner's friend scenario has a consistent resolution:*

(i) The friend's context C_F and Wigner's context C_W are distinct objects of $\mathscr{C}$ with no common refinement that would force agreement on the "state" of the system.

(ii) From C_F, the presheaf assigns a definite outcome: $S(C_F) \ni \psi_\lambda$ for some eigenvalue λ.

(iii) From C_W, the presheaf assigns a superposition: $S(C_W) \ni \sum_\lambda c_\lambda(\psi_\lambda \otimes |F_\lambda\rangle)$, where $|F_\lambda\rangle$ encodes the friend's state correlated with outcome λ.

(iv) There is no contradiction because C_F and C_W are complementary contexts: they access different sections of the same *presheaf S, and the presheaf is perfectly consistent in assigning different data to different contexts.*

(v) Interference experiments by Wigner (acting on the joint system from C_W) can in principle reveal the coherence that is invisible from C_F, confirming that the presheaf is richer than any single context.

Proof. The key is that C_F and C_W are not refinements of each other. The friend's measurement selects a morphism $f_F : C_{\text{lab}} \to C_F$, restricting the presheaf to the friend's perspective. Wigner's description uses a different morphism $f_W : C_{\text{lab}} \to C_W$, which does not factor through C_F (since Wigner treats the friend as a quantum system, not as a classical observer).

Formally:

- $S(C_F)$: The fiber at the friend's context. Here, $S(C_F)$ contains the eigenstates of the measured observable, and the friend accesses one of them.
- $S(C_W)$: The fiber at Wigner's context. Here, $S(C_W) \cong S_{\text{system}} \otimes S_{\text{friend}}$, and the composite state is an entangled superposition.

These are not contradictory descriptions of a single reality but different evaluations of the presheaf at different contexts. The consistency condition is satisfied: there exists a morphism $g : C_W \to C_F$ (Wigner can "ask the friend"), and $S(g)$ maps the entangled state in $S(C_W)$ to the definite state in $S(C_F)$ by performing the appropriate partial trace/projection.

The apparent paradox dissolves because the question "is the state definite or superposed?" is context-dependent: the answer is "definite from C_F" and "superposed from C_W," and these answers are simultaneously true aspects of the single presheaf S. □

6 Bell's Theorem and Contextuality

6.1 Bell's Theorem as a Categorical Structure Theorem

Bell's theorem [7] demonstrates that no local hidden variable theory can reproduce the predictions of quantum mechanics. In the categorical framework, this becomes a theorem about the non-existence of certain natural transformations.

Theorem 6.1 (Categorical Bell Theorem)**.** *Let S_{12} be an entangled presheaf on $\mathscr{C} \times \mathscr{C}$. There exists no natural transformation*

$$\lambda : S_{12} \Rightarrow S_1 \boxtimes S_2 \tag{36}$$

that simultaneously:

(i) is a section of the projection $S_1 \boxtimes S_2 \to S_{12}$ (determinism);

(ii) factors through representable presheaves at each context (locality);

(iii) preserves the inner product structure (physicality).

Proof. Suppose such a λ exists. Then for each pair (C_1, C_2), we have a map $\lambda_{(C_1,C_2)}$: $S_{12}(C_1, C_2) \to S_1(C_1) \otimes S_2(C_2)$ that factors as $\lambda_{C_1} \otimes \lambda_{C_2}$ (by the locality condition). This means every state in $S_{12}(C_1, C_2)$ maps to a product state in $S_1(C_1) \otimes S_2(C_2)$.

But an entangled state $\Psi \in S_{12}(C_1, C_2)$ has Schmidt rank $r > 1$, so Ψ cannot be expressed as a single product vector $\psi_1 \otimes \psi_2$. The map $\lambda_{C_1} \otimes \lambda_{C_2}$ applied to Ψ would produce a product vector, which cannot equal Ψ (since Ψ is not a product vector and λ is required to be a section of the projection, hence the identity on the image).

Therefore, no such λ exists. The local hidden variable assumption (factorization through representable presheaves at each context independently) is incompatible with entanglement. □

6.2 The CHSH Inequality

The CHSH inequality [11] provides a quantitative test of Bell's theorem.

Theorem 6.2 (CHSH Inequality and Tsirelson Bound). *Let A_1, A_2 be observables on system 1 and B_1, B_2 observables on system 2, all with eigenvalues ± 1. Define the CHSH operator*

$$\mathcal{B} = A_1 \otimes B_1 + A_1 \otimes B_2 + A_2 \otimes B_1 - A_2 \otimes B_2. \tag{37}$$

Then:

(i) ***Local hidden variable bound:*** *If a factored natural transformation λ as in Theorem 6.1 exists, then $|\langle \mathcal{B} \rangle| \leq 2$.*

(ii) ***Quantum bound (Tsirelson):*** *In the presheaf framework, $|\langle \mathcal{B} \rangle| \leq 2\sqrt{2}$.*

(iii) ***Saturation:*** *The bound $2\sqrt{2}$ is achieved by the Bell state $|\Phi^+\rangle$ with suitable choices of A_i, B_j.*

Proof. (i) If λ exists, then for each hidden variable value μ, the observables have definite values $a_i(\mu) = \pm 1$ and $b_j(\mu) = \pm 1$. Direct calculation shows $a_1 b_1 + a_1 b_2 + a_2 b_1 - a_2 b_2 = a_1(b_1 + b_2) + a_2(b_1 - b_2)$. Since $b_1, b_2 = \pm 1$, either $b_1 + b_2 = 0$ or $b_1 - b_2 = 0$, so $|a_1(b_1 + b_2) + a_2(b_1 - b_2)| \leq 2$. Averaging over μ gives $|\langle \mathcal{B} \rangle| \leq 2$.

(ii) The operator $\mathcal{B}$ satisfies $\mathcal{B}^2 = 4I - [A_1, A_2] \otimes [B_1, B_2]$. Since $\|[A_i, A_j]\| \leq 2\|A_i\|\|A_j\| = 2$ for ± 1-valued operators, we have $\|\mathcal{B}^2\| \leq 4+4 = 8$, giving $\|\mathcal{B}\| \leq 2\sqrt{2}$.

(iii) Choose $A_1 = \sigma_z$, $A_2 = \sigma_x$, $B_1 = (\sigma_z + \sigma_x)/\sqrt{2}$, $B_2 = (\sigma_z - \sigma_x)/\sqrt{2}$. Then $\langle \Phi^+ | \mathcal{B} | \Phi^+ \rangle = 2\sqrt{2}$.

The categorical interpretation is that the violation of the CHSH inequality witnesses the non-separability of the presheaf S_{12}: the correlations accessible from the product context (C_1, C_2) exceed what any factored presheaf can produce, precisely because S_{12} is an irreducible presheaf on $\mathscr{C} \times \mathscr{C}$. □

6.3 The Kochen–Specker Theorem as Non-Existence of Global Sections

Theorem 6.3 (Categorical Kochen–Specker). *For* $\dim \mathcal{H} \geq 3$, *there exists no global section of the presheaf of valuations* : $\mathscr{C}^{\mathrm{op}} \to \{$ *definedby*$(C) = \{v : \mathcal{O}(C) \to \{0, 1\} \mid v$ *preserves functional relations*$\}$,(38) where $\mathcal{O}(C)$ is the set of observables compatible with context C. That is, $\varprojlim_C(C) = \emptyset$.

Proof. A global section would assign to each projection operator P a value $v(P) \in \{0, 1\}$ such that:

(i) $v(I) = 1$;

(ii) for any resolution of identity $\sum_i P_i = I$ by orthogonal projections, $\sum_i v(P_i) = 1$.

This is a non-contextual hidden variable assignment. The Kochen–Specker theorem [12] shows that no such assignment exists in $\dim \mathcal{H} \geq 3$ by constructing an explicit finite set of projections for which conditions (i) and (ii) are contradictory.

Categorically, this means the presheaf has no global section: local valuations at individual contexts C exist (each context, being a commutative subalgebra, admits classical valuations), but these local sections cannot be glued into a global section because the compatibility conditions across non-commutative contexts are inconsistent.

This is precisely the statement that is not a *sheaf* with respect to any Grothendieck topology that covers all contexts. The obstruction is cohomological: the first cohomology group $H^1(\mathscr{C},)$ is non-trivial, encoding the contextuality of quantum mechanics. □

Remark 6.4 (Contextuality as Cohomological Obstruction). The failure of the presheaf to be a sheaf is measured by its sheafification defect. Following Abramsky and Brandenburger [20], contextuality can be characterized as a cohomological obstruction: the existence of non-trivial elements in $H^1(\mathscr{C},)$ (or higher cohomology groups) quantifies the degree to which local valuations fail to globalize. This connects the categorical framework to the sheaf-cohomological approach to contextuality.

7 Quantum Teleportation and Entanglement Swapping

7.1 Quantum Teleportation as Perspective Transfer

Quantum teleportation [13] is the transfer of quantum information using entanglement and classical communication. In the categorical framework, it becomes a *transfer of perspective*.

Definition 7.1 (Teleportation Protocol)**.** The quantum teleportation protocol consists of:

(i) **Resource:** An entangled Bell state $|\Phi^+\rangle_{BC}$ shared between parties B (Alice) and C (Bob), represented as a maximally entangled presheaf on $\mathscr{C}_B \times \mathscr{C}_C$.

(ii) **Input:** An unknown state $|\psi\rangle_A$ held by Alice, represented as a section of a qubit presheaf Q_A.

(iii) **Bell measurement:** Alice performs a joint measurement on systems A and B in the Bell basis, corresponding to a morphism $f_{\mathrm{Bell}} : C^A_{\mathrm{lab}} \to C_{\mathrm{Bell}}$ in $\mathscr{C}_A \times \mathscr{C}_B$.

(iv) **Classical communication:** Alice communicates her Bell measurement outcome $k \in \{1, 2, 3, 4\}$ to Bob.

(v) **Correction:** Bob applies a unitary U_k (one of $\{I, \sigma_x, \sigma_z, i\sigma_y\}$) to his qubit, corresponding to a natural automorphism of Q_C.

Theorem 7.2 (Teleportation as Presheaf Isomorphism)**.** *The teleportation protocol induces a natural isomorphism between the presheaf data at Alice's input context and the presheaf data at Bob's output context:*

$$Q_C(C_{\mathrm{out}}) \xrightarrow{\sim} Q_A(C_{\mathrm{in}}), \tag{39}$$

mediated by the entangled presheaf S_{BC} and the Bell measurement.

Proof. The initial state of the three-qubit system is

$$|\Psi_0\rangle = |\psi\rangle_A \otimes |\Phi^+\rangle_{BC} = (\alpha\,|0\rangle_A + \beta\,|1\rangle_A) \otimes \frac{1}{\sqrt{2}}(|00\rangle_{BC} + |11\rangle_{BC}). \tag{40}$$

Expanding in the Bell basis $\{|\Phi^\pm\rangle, |\Psi^\pm\rangle\}$ on the AB subsystem:

$$\begin{aligned}|\Psi_0\rangle = \frac{1}{2}\Big[& |\Phi^+\rangle_{AB}(\alpha\,|0\rangle_C + \beta\,|1\rangle_C) + |\Phi^-\rangle_{AB}(\alpha\,|0\rangle_C - \beta\,|1\rangle_C) \\ & + |\Psi^+\rangle_{AB}(\alpha\,|1\rangle_C + \beta\,|0\rangle_C) + |\Psi^-\rangle_{AB}(-\alpha\,|1\rangle_C + \beta\,|0\rangle_C)\Big].\end{aligned} \tag{41}$$

Alice's Bell measurement selects one of the four terms. Categorically, this is a restriction of the three-system presheaf S_{ABC} along the Bell measurement morphism f_{Bell}:

$$S_{ABC}(f_{\text{Bell}}) : S_{ABC}(C_A \times C_B \times C_C) \to S_{ABC}(C_{\text{Bell}} \times C_C). \tag{42}$$

After Alice obtains outcome k and communicates it, Bob applies U_k, which is a natural automorphism $U_k : Q_C \Rightarrow Q_C$. The result is:

$$U_k(\text{Bob's state after outcome } k) = \alpha\,|0\rangle_C + \beta\,|1\rangle_C = |\psi\rangle_C\,. \tag{43}$$

The presheaf data $|\psi\rangle$ has been "transferred" from $Q_A(C_{\text{in}})$ to $Q_C(C_{\text{out}})$, but no physical object has moved. What has changed is the *perspective*: the data that was accessible via the context C_A is now accessible via the context C_C, mediated by the entangled resource S_{BC} and the Bell measurement.

The no-signaling theorem is preserved because without Alice's classical communication (specifying k), Bob's reduced state is the maximally mixed state $I/2$, containing no information about $|\psi\rangle$. Categorically, the reduced presheaf $\text{Tr}_{AB}(S_{ABC})$ at Bob's context is independent of $|\psi\rangle$. □

Remark 7.3 (No-Cloning and Teleportation). Teleportation does not clone the state $|\psi\rangle$. Alice's Bell measurement destroys the coherence of $|\psi\rangle_A$—after measurement, system A is in a Bell state with B, not in $|\psi\rangle$. Categorically, the presheaf section at C_A has changed from $|\psi\rangle$ to a component of a Bell state. This is consistent with the no-cloning theorem (Corollary 3.17).

7.2 Entanglement Swapping

Entanglement swapping is the process by which two particles that have never interacted become entangled, mediated by a Bell measurement on intermediate particles.

Definition 7.4 (Entanglement Swapping Protocol)**.** Given two entangled pairs—$|\Phi^+\rangle_{AB}$ shared between Alice and Bob, and $|\Phi^+\rangle_{CD}$ shared between Bob and Charlie—Bob performs a Bell measurement on his two particles B and C. After the measurement, particles A (Alice) and D (Charlie) become entangled.

Theorem 7.5 (Entanglement Swapping as Presheaf Composition)**.** *Entanglement swapping is the categorical composition of entangled presheaves via a joint measurement:*

$$S_{AB} \circ_B S_{CD} \xrightarrow{\textit{Bell meas.}} S_{AD}. \tag{44}$$

The resulting presheaf S_{AD} is maximally entangled on $\mathscr{C}_A \times \mathscr{C}_D$.

Proof. The initial state of the four-qubit system is

$$|\Psi_0\rangle = |\Phi^+\rangle_{AB} \otimes |\Phi^+\rangle_{CD} = \frac{1}{2}(|00\rangle_{AB} + |11\rangle_{AB}) \otimes (|00\rangle_{CD} + |11\rangle_{CD}). \tag{45}$$

Rewriting in the Bell basis for the BC subsystem:

$$\begin{aligned}|\Psi_0\rangle = \frac{1}{2}\Big[& |\Phi^+\rangle_{BC} \otimes |\Phi^+\rangle_{AD} + |\Phi^-\rangle_{BC} \otimes |\Phi^-\rangle_{AD} \\ & + |\Psi^+\rangle_{BC} \otimes |\Psi^+\rangle_{AD} + |\Psi^-\rangle_{BC} \otimes |\Psi^-\rangle_{AD}\Big].\end{aligned} \tag{46}$$

Bob's Bell measurement on BC selects one of the four terms. In each case, A and D are left in a Bell state—a maximally entangled state—despite having never interacted.

Categorically, this is a composition of presheaves. The entangled presheaf S_{AB} correlates contexts in $\mathscr{C}_A$ with contexts in $\mathscr{C}_B$, and S_{CD} correlates $\mathscr{C}_C$ with $\mathscr{C}_D$. Bob's Bell measurement creates a morphism $\mathscr{C}_B \times \mathscr{C}_C \to \mathscr{C}_{\text{Bell}}^{BC}$ that "fuses" the B and C context categories. The composition through this fusion produces a new presheaf on $\mathscr{C}_A \times \mathscr{C}_D$ that is non-separable:

$$S_{AD} = S_{AB} \circ_{\text{Bell}} S_{CD}, \tag{47}$$

where $\circ_{\text{Bell}}$ denotes composition mediated by the Bell measurement.

The presheaf S_{AD} is maximally entangled because the Schmidt coefficients of the resulting Bell state are equal ($1/\sqrt{2}$). The non-locality of entanglement swapping is not "spooky action at a distance" but the creation of an irreducible joint relational structure through the composition of presheaves. □

Remark 7.6 (Entanglement Swapping and the Yoneda Lemma). The Yoneda Lemma provides the ultimate justification for entanglement swapping: particles A and D become entangled because their joint relational profile (the presheaf S_{AD}) is non-separable. The relational profile is determined not by direct interaction but by the composition of morphisms through the intermediary BC. This is entirely analogous to how, in pure category theory, the composition $\text{Hom}(A, B) \times \text{Hom}(B, C) \to \text{Hom}(A, C)$ creates direct relationships via composition through an intermediary.

7.3 Quantum Dense Coding

For completeness, we include the categorical treatment of superdense coding, the dual protocol to teleportation.

Proposition 7.7 (Superdense Coding as Dual Perspective Transfer)**.** *By sharing a Bell state $|\Phi^+\rangle_{AB}$ and transmitting one qubit, Alice can communicate two classical bits to Bob. Categorically, Alice applies one of four natural automorphisms $\{I, \sigma_x, \sigma_z, i\sigma_y\}$ to her qubit presheaf Q_A, transforming $|\Phi^+\rangle$ into one of the four Bell states. Bob's Bell measurement on the joint system $Q_A \boxtimes Q_B$ then determines which automorphism Alice applied. The presheaf on $\mathscr{C}_A \times \mathscr{C}_B$ carries the information of two classical bits via its four orthogonal maximally entangled sections.*

8 Haskell Implementation

We provide a Haskell implementation that computationally verifies the key categorical structures described in this paper. The implementation models presheaves, product categories, entangled states, Schmidt decomposition, the CHSH inequality, and quantum teleportation.

The implementation consists of the following modules:

- `Main.hs`: Top-level driver that runs all verification tests.
- Core categorical structures: categories, presheaves, product categories.
- Entanglement module: Bell states, Schmidt decomposition, entanglement entropy, monogamy verification.
- Complementarity module: non-commutative contexts, uncertainty relation verification.
- Measurement module: perspective selection, decoherence, CHSH inequality.
- Teleportation module: teleportation protocol, entanglement swapping.

Key type signatures include:

```
-- Presheaf on a category C valued in Hilb
type Presheaf c = c -> HilbertSpace

-- Composite presheaf on product category
type CompositePSh c = (c, c) -> HilbertSpace

-- Entangled state: non-separable element
data EntangledState = EntangledState
  { schmidtCoeffs :: [Double]
  , schmidtRank   :: Int
  , entropy       :: Double
  }

-- Bell measurement outcome
data BellOutcome = PhiPlus | PhiMinus | PsiPlus | PsiMinus
```

The complete implementation is provided in the accompanying source file `src/entanglement-co`
Running the code produces verification output confirming:

1. Bell states have Schmidt rank 2 and maximal entanglement entropy $\log 2$.
2. The CHSH inequality is violated up to $2\sqrt{2} \approx 2.828$.
3. Quantum teleportation faithfully transmits state data.
4. Entanglement swapping produces maximally entangled output states.
5. The monogamy inequality is satisfied for sample three-qubit states.
6. Mutually unbiased bases satisfy the uniform overlap condition.

9 Discussion and Open Problems

9.1 Summary of Results

We have developed a comprehensive categorical framework in which the three pillars of quantum phenomenology—entanglement, complementarity, and measurement—emerge as structural consequences of the Yoneda Constraint. The key results are:

1. **Entanglement** is non-separability of presheaves on product categories (Definitions 3.3–3.4). The Schmidt decomposition (Theorem 3.6) is a categorical decomposition, and the monogamy of entanglement (Theorem 3.16) is a sheaf-like condition.

2. **Complementarity** arises from non-commutativity of contexts (Theorem 4.3). The Heisenberg uncertainty relation (Theorem 4.4) is forced by the non-existence of common refinements, and mutually unbiased bases (Theorem 4.7) correspond to maximal complementarity.

3. **Measurement** is perspective selection (Definition 5.1). There is no collapse (Theorem 5.3), decoherence arises from coarse-graining (Theorem 5.5), and the measurement problem dissolves (Theorem 5.7).

4. **Bell's theorem** (Theorem 6.1) and the **Kochen–Specker theorem** (Theorem 6.3) are categorical structure theorems about the non-existence of certain natural transformations and global sections, respectively.

5. **Quantum teleportation** (Theorem 7.2) and **entanglement swapping** (Theorem 7.5) are categorical compositions of presheaves via Bell measurements.

9.2 Relation to Prior Work

The topos-theoretic approach to quantum mechanics initiated by Isham and Butterfield [16, 17] and developed by Döring and Isham [18] shares our emphasis on presheaves and contextuality. Our contribution is to unify entanglement, complementarity, and measurement under a single categorical umbrella, deriving all three from the Yoneda Constraint.

The categorical quantum mechanics of Abramsky and Coecke [19] works within compact closed categories and provides string diagrammatic tools for quantum protocols. Our framework is complementary: we derive the categorical structures from the Yoneda Constraint, while CQM takes them as given and develops their compositional calculus. The integration of these two approaches—using our framework to justify the structures and CQM to compute with them—is a promising direction.

The sheaf-cohomological approach to contextuality by Abramsky and Brandenburger [20] is directly connected to our Kochen–Specker analysis (Theorem 6.3). Our contribution is to embed this within the broader Quantum Perspectivism framework, showing that contextuality is one manifestation of the general principle that presheaves encode more than any single context can access.

9.3 Open Problems

Higher categorical structure. The framework presented here uses ordinary (1-)categories. Higher categorical generalizations—using 2-categories, ∞-categories, or $(\infty, 1)$-topoi—may reveal additional structure, particularly for the treatment of gauge symmetries and topological quantum field theories.

Continuous variable systems. Our treatment has focused primarily on finite-dimensional systems. Extending to infinite-dimensional Hilbert spaces (continuous variable quantum mechanics) requires care with the categorical constructions, particularly the treatment of unbounded operators and the nuclear spectral theorem.

Quantitative contextuality measures. While we have identified contextuality as a cohomological obstruction (Remark following Theorem 6.3), developing quantitative measures of contextuality within the presheaf framework—and connecting them to operational advantages in quantum computation—is an important open problem.

Relativistic entanglement. Extending the categorical treatment of entanglement to relativistic settings, where the context category $\mathscr{C}$ includes Lorentzian causal structure, would connect this work to algebraic quantum field theory and provide insights into the black hole information paradox.

Experimental predictions. A key question is whether the Quantum Perspectivism framework makes predictions that differ from standard quantum mechanics. Potential avenues include: novel constraints on multi-party entanglement from presheaf cohomology, predictions for the quantum-to-classical transition based on the structure of $\mathscr{C}_{\text{macro}}$, and constraints on quantum gravity from the requirement that $\mathscr{C}$ be self-consistent.

10 Conclusion

We have shown that entanglement, complementarity, and measurement—the three defining phenomena of quantum mechanics—are not independent features requiring separate explanations but manifestations of a single categorical principle: the Yoneda Constraint. Entanglement is the non-separability of presheaves on product categories, reflecting the existence of irreducibly joint relational structure. Complementarity is the non-commutativity of context morphisms, reflecting the impossibility of simultaneously refining certain pairs of observational perspectives. Measurement is the selection of a perspective, reflecting the evaluation of a presheaf at a particular context.

The framework dissolves the measurement problem (there is no collapse, only perspective selection), explains Bell inequality violations (entangled presheaves exceed the correlations possible for factored presheaves), derives the uncertainty principle from category structure (non-existence of common refinements), and provides natural categorical formulations of quantum information protocols (teleportation as perspective transfer, entanglement swapping as presheaf composition).

The deepest lesson is that quantum mechanics is not mysterious if we take seriously the Yoneda Lemma's message: objects are constituted by their relationships. A composite system has relationships that transcend its parts (entanglement), some relationships cannot coexist (complementarity), and accessing a particular relationship is a matter of choosing a vantage point (measurement). Quantum mechanics is the physics of relational structure, and the Yoneda Lemma is its constitution.

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